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A sufficient condition for the convergence of the inexact Uzawa algorithm for saddle point problems ☆

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Abstract

In this paper, the convergence property of the inexact Uzawa algorithm for solving symmetric indefinite linear systems is studied. A simple sufficient condition for the convergence of the inexact Uzawa algorithm is obtained. Two examples and numerical experiments illustrating the conclusion are provided. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper provides convergence analysis for the inexact Uzawa method applied to the solution of linear saddle point systems. Such systems are obtained when we want to solve problems with equality constraints [9], mixed formulation of second-order elliptic problems [14], the equations of elasticity and Stokes in fluid mechanics, the incompressible miscible or immiscible problems of oil and water [6,7,10]. As the mesh is refined, the resulting discrete systems can become very large, therefore, it is of great interest to develop efficient iterative methods for such problems. A method that has been frequently used is known as Uzawa algorithm, and the convergence of Uzawa algorithm was studied in many papers [2–5,13]. There are also many related papers on the preconditioned iterative methods for saddle point problems [1,11,15]. In [2], the inexact Uzawa algorithm was studied, bounds for the rates of convergence were provided. The inexact Uzawa algorithms are simple to implement and

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require minimal computer memory so they are widely used in scientific computing. In this paper, we study the eigenvalues of the iterative matrix of the inexact Uzawa algorithm. We will show that the inexact Uzawa algorithm converges provided that the preconditioners defining the algorithm satisfy one simple condition. At the end of this paper, this new sufficient condition is illustrated by two examples and numerical experiments.

2. Inexact Uzawa algorithm

Consider the linear systems of equations:

$$\begin{pmatrix} A & B^{\mathrm{T}} \\ B & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}, \tag{2.1}$$

where $F \in H_1$ and $G \in H_2$ are given and $X \in H_1$ and $Y \in H_2$ are the unknowns. Here, H_1 and H_2 are finite dimensional Hilbert spaces with inner products which we shall denote by $(\cdot,\cdot)$. $A: H_1 \mapsto H_1$ is a linear, symmetric and positive definite operator, $B: H_1 \mapsto H_2$ is a linear map and $B^{\mathrm{T}}: H_2 \mapsto H_1$ is its adjoint operator. We assume that (2.1) has one unique solution. Let preconditioners $Q_A: H_1 \mapsto H_1$, $Q_B: H_2 \mapsto H_2$ be linear, symmetric and positive definite operators. In practice, the above operators are often matrices [2,8], and we consider them as matrices in the following. The inexact Uzawa algorithm in a general form is defined as follows:

Algorithm 2.1. *For $X_0 \in H_1$ and $Y_0 \in H_2$ be given, the sequence $\{X_i, Y_i\}$ is defined for $i = 1, 2\ldots$ by*

$$\begin{cases} X_{i+1} = X_i + Q_A^{-1}(F - (AX_i + B^{\mathrm{T}} Y_i)), \\ Y_{i+1} = Y_i + Q_B^{-1}(BX_{i+1} - G). \end{cases} \tag{2.2}$$

We obtain the preconditioned Uzawa algorithm when $Q_A = A$ and Uzawa algorithm when $Q_A = A$ and $Q_B = \tau I$ (τ is a given real number, and I is the identity operator in H_1). Let $\{X, Y\}$ be the exact solution for (2.1) and $\{X_i, Y_i\}$ be the ith iteration of the inexact Uzawa algorithm, and let

$$e_i = \begin{pmatrix} X - X_i \\ Y - Y_i \end{pmatrix}, \quad M = \begin{pmatrix} I - Q_A^{-1}A & -Q_A^{-1}B^{\mathrm{T}} \\ Q_B^{-1}B(I - Q_A^{-1}A) & I - Q_B^{-1}BQ_A^{-1}B^{\mathrm{T}} \end{pmatrix}, \tag{2.3}$$

then e_i are the associate error vectors, and e_{i+1} satisfy the following equality

$$e_{i+1} = \begin{pmatrix} I - Q_A^{-1}A & -Q_A^{-1}B^{\mathrm{T}} \\ Q_B^{-1}B(I - Q_A^{-1}A) & I - Q_B^{-1}BQ_A^{-1}B^{\mathrm{T}} \end{pmatrix} e_i = Me_i. \tag{2.4}$$

As it is well-known, the error vectors of these iterative methods tends to zero vector for all e_0 if and only if the spectral radius $\rho(M)$ of the matrix M is less than unity [16]. In this paper, we analyze the spectrum of matrix M and give a general relationship between Q_A, Q_B, A and B. Certain assumptions will be imposed upon preconditioners Q_A and Q_B. The condition is sufficient for the convergence of the inexact Uzawa algorithm.

3. Convergence condition for the iteration

We state our main result in the following theorem:

Theorem 3.1. *Assume that the preconditioners Q_A and Q_B in the inexact Uzawa algorithm* (2.2) *are linear, symmetric and positive definite, then the inexact Uzawa algorithm always converges provided that the preconditioners satisfy the following condition*:

$$4Q_A - 2A - B^{\mathrm{T}}Q_B^{-1}B \quad \text{is positive definite.} \tag{3.1}$$

Proof. Assume that λ be one of the eigenvalues of M with corresponding eigenvector $\{u, v\}$, i.e.,

$$\begin{cases} (I - Q_A^{-1}A)u - Q_A^{-1}B^{\mathrm{T}}v = \lambda u, \\ Q_B^{-1}B(I - Q_A^{-1}A)u + (I - Q_B^{-1}BQ_A^{-1}B^{\mathrm{T}})v = \lambda v. \end{cases} \tag{3.2}$$

Substitute $(I - Q_A^{-1}A)u$ in the first equation of (3.2) into the second one gives

$$(1-\lambda)v = -\lambda Q_B^{-1}Bu. \tag{3.3}$$

Multiply the first equation of (3.2) by $(1-\lambda)$ and substitute (3.3) into it yields

$$[(\lambda - 1)^2 Q_A - (1-\lambda)A + \lambda B^{\mathrm{T}}Q_B^{-1}B]u = 0. \tag{3.4}$$

The above equality is fundamental in the analysis that follows. We will seek a sufficient condition for $|\lambda| < 1$.

First, we prove that $u \neq 0$ (0 is the zero vector, may differ in different spaces). In fact, on the contrary, assume that $u = 0$, since $\{u, v\}$ is an eigenvector, v cannot be a zero vector. From (3.3) we can draw the conclusion that $\lambda = 1$. Putting $\lambda = 1$ into (3.2), we find that the following equalities hold:

$$\begin{cases} Au + B^{\mathrm{T}}v = 0, \\ BQ_A^{-1}[(Q_A - A)u - B^{\mathrm{T}}v] = 0. \end{cases} \tag{3.5}$$

Combine the first equation of (3.5) with the second one gives

$$\begin{cases} Au + B^{\mathrm{T}}v = 0, \\ Bu = 0, \end{cases}$$

so the primary problem (2.1) has nonzero solution when $F = G = 0$. This contradiction shows that any eigenvector $\{u, v\}$ of M must have a nonzero component u.

Second, we consider the case when $Bu = 0$. From (3.3), in a similar manner we can prove $\lambda \neq 1$, therefore $v = 0$ and consequently $[(1-\lambda)Q_A - A]u = 0$. Taking an inner product with u yields $\lambda = 1 - (Au, u)/(Q_A u, u)$. Since (3.1) now gives $((4Q_A - 2A)u, u) > 0$, we conclude that $|\lambda| < 1$.

Finally, we assume $Bu \neq 0$. Taking an inner product with u in (3.4), it follows that:

$$(\lambda - 1)^2(Q_A u, u) - (1 - \lambda)(Au, u) + \lambda(B^{\mathrm{T}} Q_B^{-1} Bu, u) = 0. \tag{3.6}$$

Note that (3.6) can be written as follows:

$$(Q_A u, u)\lambda^2 + [(Au, u) + (Q_B^{-1} Bu, Bu) - 2(Q_A u, u)]\lambda + ((Q_A - A)u, u) = 0. \tag{3.7}$$

According to Theorem 6.2 of [12], we know that the sufficient and necessary condition for the modules of the two roots of the quadratic polynomial $\lambda^2 + p\lambda + q = 0$ be less than unity is that: $|p| < 1+q < 2$. Since $u \neq 0$, we get $(Q_A u, u) \neq 0$, therefore, that the module of each root of equation (3.7) is less than unity is equivalent to

$$|((A - 2Q_A + B^{\mathrm{T}} Q_B^{-1} B)u, u)| < ((2Q_A - A)u, u) < 2(Q_A u, u), \tag{3.8}$$

which leads to

$$((4Q_A - 2A - B^{\mathrm{T}} Q_B^{-1} B)u, u) > 0, \tag{3.9}$$

where u is the first block component of an eigenvector of M. Therefore, (3.1) is a sufficient condition for $|\lambda| < 1$. Hence, the convergence of the inexact Uzawa algorithm is assured. This completes the proof of the theorem. □

Remark 3.1. In the above analysis, we can easily see that (3.9) is the sufficient and necessary condition for the inexact Uzawa algorithm to converge. If the set of all the first part of the eigenvectors of M (i.e. u) contains a basis of H_1, then (3.9) becomes both sufficient and necessary.

Remark 3.2. The condition (3.1) is weaker than the conditions in [2], because we do not demand that the condition (3.2) in [2] should be satisfied. This is justified in Example 4.1 and the numerical experiments given in Table 1.

If we make the following hypotheses: there exist positive constants a and b, such that

$$a(Aw, w) \leqslant (Q_A w, w), \quad \forall w \in H_1, \tag{3.10}$$

$$(Q_B^{-1} Bw, Bw) \leqslant b(Aw, w), \quad \forall w \in H_1. \tag{3.11}$$

Using the above assumptions on Q_A, Q_B, we can give a sufficient condition for (3.1):

$$4a - 2 - b > 0. \tag{3.12}$$

Remark 3.3. From the well-known Courant–Fischer min–max theorem [17], $1/a$ and b can be viewed as the upper bounds of the numerical radii of $Q_A^{-1/2} A Q_A^{-1/2}$ and $A^{-1/2} B^{\mathrm{T}} Q_B^{-1} B A^{-1/2}$, respectively, and the condition (3.12) is the constraint for the convergence of the inexact Uzawa algorithm.

4. Examples

In this section, we verify that the condition for convergence derived above coincides with some known results. The examples show that condition (3.1) or (3.12) is not only sufficient, but also nearly necessary.

Example 4.1. First, let $Q_A = cA$ (without the assumption that $c \geqslant 1$) and $Q_B^{-1} = \tau I$ (scalars $c, \tau > 0$). Under this circumstance, we can put $a = c$ and the condition (3.12) becomes

$$b < 4c - 2. \tag{4.1}$$

Note that $(BA^{-1/2})^{\mathrm{T}} BA^{-1/2}$ is a symmetric matrix, therefore, for all $w \in H_1$, $w \neq 0$, we have

$$\sup_{w \in H_1} \frac{(Bw, Bw)}{(Aw, w)} = \sup_{w \in H_1} \frac{(BA^{-1/2}w, BA^{-1/2}w)}{(w, w)} = \sup_{w \in H_1} \frac{((BA^{-1/2})^{\mathrm{T}} BA^{-1/2}w, w)}{(w, w)}$$

$$= \lambda_{\max}[(BA^{-1/2})^{\mathrm{T}} BA^{-1/2}] = \lambda_{\max}[BA^{-1}B^{\mathrm{T}}], \tag{4.2}$$

where the notation $\lambda_{\max}[M_1]$ is the largest eigenvalue of the matrix M_1. In the derivation above, we have used the properties of the Rayleigh quotient and that for $M_2 \in R^{m \times n}$, $M_3 \in R^{n \times m}$, the matrix $M_2 M_3$ has the same nonzero eigenvalues as that of the matrix $M_3 M_2$ [17]. Thus, we have

$$\tau \lambda_{\max}[BA^{-1}B^{\mathrm{T}}] \leqslant b < 4c - 2. \tag{4.3}$$

If $c = 1$, algorithm (2.2) is the exact Uzawa algorithm, (4.3) coincides with the conclusion given in [8] (here C is a zero matrix). Here (4.1) is also a necessary condition.

Example 4.2. This example is stimulated by the example given in [1]. Now we let $B = \tau A^{1/2}$, $Q_A = \omega A$, $Q_B = \theta I$, $\tau, \theta > 0$, $\omega = 1 - \alpha$ $(0 < \alpha < 1)$. The special case when $\theta = \tau = 1$ appeared in [1]. The matrix M has the following form:

$$M = \begin{pmatrix} (1 - \frac{1}{\omega})I & -\frac{\tau}{\omega}A^{-1/2} \\ \frac{\tau}{\theta}(1 - \frac{1}{\omega})A^{1/2} & (1 - \frac{\tau^2}{\theta\omega})I \end{pmatrix}.$$

According to our analysis above, we can now let $a = \omega$, $b = \tau^2/\theta$. If

$$4\omega - 2 - \frac{\tau^2}{\theta} > 0, \tag{4.4}$$

then the spectral radius of M is less than one.

In fact, considering that $0 < \omega < 1$, the 2×2 block matrix M has only the two real eigenvalues

$$\lambda_{1,2} = \frac{(2\omega - 1 - \frac{\tau^2}{\theta}) \pm \sqrt{(1 + \frac{\tau^2}{\theta})^2 - 4\omega\frac{\tau^2}{\theta}}}{2\omega},$$

then $\rho(M) < 1$ is equivalent to $|2\omega - 1 - \tau^2/\theta \pm \sqrt{(1 + \tau^2/\theta)^2 - 4\omega\tau^2/\theta}| < 2\omega$, that is to say, $1 + \tau^2/\theta - 4\omega + \sqrt{(1 + \tau^2/\theta)^2 - 4\omega\tau^2/\theta} < 0$, which leads to condition (4.4) exactly.

Table 1
$Q_A = \alpha A$ and $Q_B = \beta I_2$

α	β	ρ	N	$\|\|e_N\|\|$
1.3719[a]	0.1	1	1000	0.0792
1.38	0.1	0.9673	343	9.8433e−7
0.5872[a]	1.0	1	1000	0.0523
0.59	1.0	0.9888	967	9.9515e−7
0.5087[a]	10.0	1	1000	0.0533
0.52	10.0	0.9797	651	9.8035e−7

[a]Stands for the α where '>' in (5.1) is replaced by '=', therefore $\rho = 1$.

Table 2
$Q_A = \gamma I_3$ and $Q_B = \delta I_2$

γ	δ	ρ	N	$\|\|e_N\|\|$
5.0	0.1	1.0332	1000	1.8903e+14
5.1	0.1	0.9556	308	9.9058e−7
2.9	1.0	1.0049	1000	114.8397
3.0	1.0	0.9329	198	9.3880e−7
2.7	10.0	1.0194	1000	1.8805e8
2.8	10.0	0.9797	650	9.8181e−7

Table 3
$Q_A = (D + L)D^{-1}(D + L)^{\mathrm{T}}$ and $Q_B = \varepsilon I_2$

ε	ρ	N	$\|\|e_N\|\|$
0.1598	1.000	1000	0.1788
0.17	0.8698	88	9.3010e−7

5. Numerical experiments

In this section, we give some numerical experiments using Matlab. Here

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and the initial value for algorithm 2.1 is $\{X_0, Y_0\} = \frac{1}{\sqrt{5}}(1, 1, 1, 1, 1)^{\mathrm{T}}$.

The stopping criterion is that $||e_i||/||e_0|| < 10^{-6}$ or $N > 1000$, here e_i is the ith residual, $||e_i||$ is the Frobenius norm of e_i. N is the number of iterations, the iterations stopped when $N > 1000$. In Tables 1, 2 and 3, α, β, γ, δ and ε are positive constants, I_{m} is the m-by-m identity matrix. In Table 3, A is decomposed into $A = D + L + U$, where $D = \mathrm{diag}(A)$, L is the lower triangular and U is the upper triangular. Q_A is given as the SSOR-PCG iteration matrix with the relaxation parameter taken to be 1.0.

The following conditions are the simplified forms of (3.1) for Tables 1, 2 and 3, respectively:

$$\alpha > \frac{1}{2} + \frac{31+\sqrt{65}}{448\beta}, \tag{5.1}$$

$$4\gamma - \frac{1}{\delta} > 10 \quad \text{and} \quad (4\gamma-8)\left[\left(4\gamma-8-\frac{1}{\delta}\right)^2 - 4\right] - 4\left(4\gamma-8-\frac{1}{\delta}\right) > 0, \tag{5.2}$$

$$8 - \frac{1}{\varepsilon} > \frac{-5+\sqrt{1321}}{18}. \tag{5.3}$$

For every value of β (or δ), there are two corresponding values of α (respectively, γ), condition (5.1) (or (5.2)) is not satisfied for the first value and it is satisfied for the second one. In Tables 1–3 are the results.

We see that the above results agree with the conclusion of Theorem 3.1.

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